

# Compatible quantum correlations: joinability and sharability properties of Werner states

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We investigate some basic scenarios in which a given set of bipartite quantum states may consistently arise as the set of reduced states of a global  $N$ -partite quantum state. Intuitively, we say that the multipartite state “joins” the underlying correlations. Determining whether, for a given set of states and a given joining structure, a compatible  $N$ -partite quantum state exist is known as the quantum marginal problem. We restrict to bipartite reduced states that belong to the paradigmatic class of Werner states in  $d$  dimensions, and focus on two specific versions of the quantum marginal problem which we find to be tractable. The first is Alice-Bob, Alice-Charlie joining, with both pairs being in a Werner state. The second is  $m$ - $n$  sharability of a Werner state across  $N$  subsystems, which may be seen as an extension of the  $N$ -representability problem to the case where the latter are partitioned into two groupings of  $m$  and  $n$  parties, respectively. By exploiting the symmetry properties that Werner states enjoy, we determine in each case necessary and sufficient conditions for arbitrary  $d$ . Our results explicitly show that although entanglement is required for sharing limitations to emerge, correlations beyond entanglement may generally suffice to restrict joinability, and not all unentangled states necessarily obey the same limitations. Implications for the joinability of arbitrary bipartite states as well as for quantum information processing tasks are discussed.

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## I. INTRODUCTION

Understanding the nature of quantum correlations in multiparty systems and the distinguishing features they exhibit relative to classical correlations is a central goal across quantum information processing (QIP) science [1], with implications ranging from condensed-matter and statistical physics to quantum chemistry, and the quantum-to-classical transition. From a foundational perspective, exploring what different kinds of correlations are, in principle, allowed by probabilistic theories more general than quantum mechanics further helps to identify under which set of physical constraints the standard quantum framework may be uniquely recovered [2, 3].

In this context, entanglement provides a distinctively quantum type of correlation, that has no analogue in classical statistical mechanics. A striking feature of entanglement is that it cannot be freely distributed among different parties: if a bipartite system, say,  $A$ (lice) and  $B$ (ob), is in a maximally entangled pure state, then no other system,  $C$ (harlie), may be correlated with it. In other words, the entanglement between  $A$  and  $B$  is *monogamous* and cannot be *shared* [4–8]. This simple tripartite setting motivates two simple questions about bipartite quantum states: given a bipartite state, we ask whether it can arise as the reduced state of  $A$ - $B$  and of  $A$ - $C$  simultaneously; or, more generally, given two bipartite states, we ask if one can arise as the reduced state of  $A$ - $B$  while the other arises as the reduced state of  $A$ - $C$ . It should be emphasized that both of these are questions about the existence of tripartite states with given reduction properties. While formal (and more general) definitions will be provided later in the paper, these examples

serve to introduce the notions of *sharing* (1-2 sharing) and *joining* (1-2 joining), respectively. In its most general formulation, the joinability problem is also known as the *quantum marginal problem* (or *local consistency problem*), which has been heavily investigated both from a mathematical-physics [9–11] and a quantum-chemistry perspective [12, 13] and is known to be QMA-hard [14]. Our choice of terminology, however, facilitates a uniform language for describing the joinability/sharability scenarios. For instance, we say that the joinable correlations of  $A$ - $B$  and  $A$ - $C$  are joined by a joining state on  $A$ - $B$ - $C$ .

The limited sharability/joinability of entanglement was first quantified in the seminal work by Coffman, Kundu, and Wootters, in terms of an exact (CKW) inequality obeyed by the entanglement across the  $A$ - $B$ ,  $A$ - $C$  and  $A$ -( $BC$ ) bipartitions, as quantified by concurrence [4]. In a similar venue, several subsequent investigations attempted to determine the extent to which different entanglement measures can be used to diagnose failures of joinability, see e.g. [7, 15, 16]. More recently, significant progress has been made in characterizing quantum correlations more general than entanglement [17, 18], in particular as captured by quantum discord [19]. While it is now established that quantum discord does not obey a monogamy inequality [20], different kind of limitations exist on the extent to which it can be freely shared and/or communicated [21, 22]. Despite these important advances, a complete picture is far from being reached. What kind of limitations do strictly mark the quantum-classical correlation boundary? What different quantum features are responsible for enforcing different aspects of such limitations, and how does this relate to the degree of resourcefulness that these correlations can have for QIP?

While the above are some of the broad questions motivating this work, our specific focus here is to make progress on joinability and sharability properties in low-dimensional multipartite settings. In this context, a recent paper [23] has obtained a necessary condition for three-party joining in terms of the subsystem entropies, and additionally established a sufficient condition in terms of the trace-norm distances between the states in question and known joinable states. To the best of our knowledge, however, no conditions that are both necessary and sufficient are available as yet. In this paper, we obtain *necessary and sufficient conditions for joinability* at the cost of losing general applicability; namely, we provide an exact characterization of both the three-party joinability and the 1- $n$  sharability in the case that each reduced bipartite state is a Werner state [24].

Though our results are restricted in scope of applicability, they provide key insights as to the sources of joinability limitations. Most importantly, we find that standard measures of quantum correlations, such as concurrence and quantum discord, do not suffice to determine the limitations in joining quantum correlations. Specifically, we find that the joined states need *not* be entangled or even discordant in order not to be joinable. Further to that, although separable states may have joinability limitations, they *are*, nonetheless, freely (arbitrarily) sharable. By introducing a one-parameter class of probability distributions we provide a natural classical analogue to the quantum Werner states. This allows us to illustrate how classical joinability restrictions carry over to the quantum case and, more interestingly, to demonstrate that the quantum case demands limitations which are not present classically. Ultimately, this feature may be traced back to complementarity of observables, which clearly plays no role in the classical case. From this point of view, it is interesting to note that the uncertainty principle was also shown to be instrumental in constraining the sharability of quantum discord [21]. It is our hope that further pursuits of more general necessary and sufficient conditions may be aided by the methods and findings herein.

The content is organized as follows. In Sec. II we introduce the relevant mathematical framework for defining joinability and sharability, and present some general preliminary results contrasting the classical and quantum cases. Sec. III contains the core results of our analysis. In particular, in Sec. IIIB we obtain necessary and sufficient conditions for joinability of three Werner states on  $d$ -dimensional subsystems (qudits), whereas in Sec. IIIC we characterize general  $m$ - $n$  sharability properties for such states. A simple analytic expression is established for the special yet important case of 1- $n$  sharability ( $m = 1$ ). In Sec. IV we describe how our Werner-state results imply *joinability constraints for arbitrary bipartite states* and briefly address some operational implications. Concluding remarks and additional open questions are presented in Sec. V, while we include in two separate Appendixes technical background on group-representation tools used in the proofs of our results.

## II. JOINING AND SHARING CLASSICAL VS. QUANTUM STATES

Although our main focus will be to quantitatively characterize simple low-dimensional settings, we introduce the relevant concepts with a higher degree of generality, in order to better highlight the underlying mathematical structure and to ease connections with existing related notions in the literature.

We are interested in the correlations among the subsystems of a  $N$ -partite composite system  $S$ . In the quantum case, we thus require a Hilbert space with a tensor product structure:

$$\mathcal{H}^{(N)} \simeq \bigotimes_{i=1}^N \mathcal{H}_i^{(1)}, \quad \dim(\mathcal{H}_i^{(1)}) \equiv d_i,$$

where  $\mathcal{H}_i^{(1)}$  represents the individual “single-particle” state spaces and, for our purposes, each  $d_i$  is finite. In the classical scenario, to each subsystem we associate a sample space  $\Omega_i$  consisting of  $d_i$  possible outcomes, with the joint sample space being given by the Cartesian product:

$$\Omega^{(N)} \simeq \Omega_1 \times \dots \times \Omega_N.$$

Probability distributions on  $\Omega^{(N)}$  are the classical counterpart of quantum density operators on  $\mathcal{H}^{(N)}$ .

### A. Joinability

The input to a joinability problem is a set of subsystem states which, in full generality, may be specified relative to a “neighborhood structure” on  $\mathcal{H}^{(N)}$  (or  $\Omega^{(N)}$ ) [25]. That is, let neighborhoods  $\{\mathcal{N}_j\}$  be given as subsets of the set of indexes labeling individual subsystems,  $\mathcal{N}_k \subsetneq \mathbb{Z}_N$ . We can then give the following:

**Definition II.1. [Quantum Joinability]** *Given a neighborhood structure  $\{\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_\ell\}$  on  $\mathcal{H}^{(N)}$ , a list of density operators  $(\rho_1, \dots, \rho_\ell) \in (\mathcal{D}(\mathcal{H}_{\mathcal{N}_1}), \dots, \mathcal{D}(\mathcal{H}_{\mathcal{N}_\ell}))$  is joinable if there exists a  $N$ -partite density operator  $w \in \mathcal{D}(\mathcal{H}^{(N)})$ , called a joining state, that reduces according to the neighborhood structure, that is,*

$$\text{Tr}_{\hat{\mathcal{N}}_k}(w) = \rho_k, \quad \forall k = 1, \dots, \ell, \quad (1)$$

where  $\hat{\mathcal{N}}_k \equiv \mathbb{Z}_N \setminus \mathcal{N}_k$  is the tensor complement of  $\mathcal{N}_k$ .

The analogous definition for classical joinability is obtained by substituting corresponding terms, in particular, by replacing the partial trace over  $\hat{\mathcal{N}}_k$  with the corresponding marginal probability distribution. As remarked, the question of joinability has been extensively investigated in the context of the classical [26] and quantum [9, 23, 27, 28] marginal problem. A joining state is equivalently referred to as an *extension* or an element of the *pre-image* of the list under the reduction map, while

the members of a list of joinable states are also said to be *compatible* or *consistent*.

Clearly, a *necessary* condition for a list of states to be joinable is that the states “agree” on any overlapping reduced states. That is, given any two states from the list whose neighborhoods are intersecting, their respective reduced states of the subsystems in the intersection must coincide. From this point of view, any failure of joinability due to a disagreement of overlapping reduced states is a trivial case of non-compatible  $N$ -party correlations. We are interested in cases where joinability fails *despite* the agreement on overlapping marginal states. Since in Sec. III we shall consider (Werner) states whose reduced states are completely mixed, this consistency requirement will be satisfied by construction.

One important feature of joinability, which has recently been investigated in [29], is the *convex* structure that both joinable states lists and joining states enjoy. In particular, the following result will be relevant:

**Proposition II.2.** *The set of lists of density operators satisfying a given joinability scenario is convex under component-wise combination.*

*Proof.* Given a joinability scenario with  $\ell$  neighborhoods, assume that  $w_A$  joins  $\vec{L}_A = (\rho_1^A, \dots, \rho_\ell^A)$  and  $w_B$  joins  $\vec{L}_B = (\rho_1^B, \dots, \rho_\ell^B)$ , respectively. An arbitrary convex combination of these lists,

$$\begin{aligned}\vec{L}_C &= \lambda \vec{L}_A + (1 - \lambda) \vec{L}_B \\ &= (\lambda \rho_1^A + (1 - \lambda) \rho_1^B, \dots, \lambda \rho_\ell^A + (1 - \lambda) \rho_\ell^B) \\ &\equiv (\rho_1^C, \dots, \rho_\ell^C),\end{aligned}$$

is then joinable by the same convex combination of joining states:

$$\begin{aligned}\text{Tr}_{\mathcal{N}_k}(w_C) &= \text{Tr}_{\mathcal{N}_k}(\lambda w_A + (1 - \lambda) w_B) \\ &= \lambda \rho_k^A + (1 - \lambda) \rho_k^B \\ &= \rho_k^C.\end{aligned}$$

Thus, convex combinations of joinable lists are joinable.  $\square$

As mentioned, one of our goals is to shed light on limitations of quantum vs. classical joinability and the extent to which entanglement may play a role in that respect. We need go no further than the simplest non-trivial joining scenario to make progress on this front. This simple scenario was already introduced in the introduction: Given two states  $\rho_{AB}$  and  $\rho_{AC}$  which have consistent reduced state on  $A$ , under which conditions can a three-partite joining state  $w_{ABC}$  be found?

That stricter joinability limitations are present in the quantum case can be immediately appreciated by considering  $\rho_{AB} = |\Psi_B\rangle\langle\Psi_B| = \rho_{AC}$ , where  $|\Psi_B\rangle$  is any maximally entangled Bell pair, in which case no  $w_{ABC}$  exists. In contrast, the following property straightforwardly holds in the classical case [26]:

**Proposition II.3.** *For any two classical probability distributions  $p(A, B)$  and  $p(A, C)$  which agree on the marginal distribution of  $A$ , there exists a compatible joint probability distribution  $w(A, B, C)$ .*

*Proof.* Let  $p(A, B)$  and  $p(A, C)$  be such that for  $a \in A$ ,  $\sum_{b \in B} p(a, b) = \sum_{c \in C} p(a, c) = p(a)$ . A valid joining probability distribution is given by

$$w(a, b, c) = \frac{p(a, b) p(a, c)}{p(a)}.$$

As noted in [23], while the above joining state is not unique, it is the one having maximal entropy, and hence it properly represents an even mixture of all valid sharing distributions.  $\square$

Therefore, any *two* consistently-overlapped classical probability distributions may be joined. Still, limitations on joining classical probability distributions *do* arise in certain joining scenarios. In general, this follows from the fact that any classical probability assignments must be consistent with some convex combination of pure states. Consider, for example, a pairwise neighborhood structure, with an associated list of states  $p(A, B)$ ,  $p(B, C)$ , and  $p(A, C)$ , which have consistent single-subsystem marginals. Clearly, if each subsystem corresponds to a bit, no convex combination of pure states gives rise to a probability distribution  $w(A, B, C)$  in which each pair is completely anticorrelated; in other words, “bits of three can’t all disagree”. In Sec. IIIB we explicitly compare this particular classical joining scenario to the analogous quantum one.

While all the classical joining limitations may be expressed by linear inequalities, the quantum joining limitations are significantly more complicated. The limitations arise from demanding that the joining operator be a valid density operator (i.e., non-negative and trace-one). Non-negativity, in fact, is a highly non-trivial requirement. This fact is demonstrated by the following proposition, which may be readily generalized to any joining scenario:

**Proposition II.4.** *For any two trace-one Hermitian operators  $Q_{AB}$  and  $Q_{AC}$  which obey the consistency condition  $\text{Tr}_B(Q_{AB}) = \text{Tr}_C(Q_{AC})$ , there exists a trace-one Hermitian joining operator  $Q_{ABC}$ .*

*Proof.* Consider an orthogonal Hermitian product basis which includes the identity for each subsystem, that is,  $\{A_i \otimes B_j \otimes C_k\}$ , where  $A_0 = B_0 = C_0 = \mathbb{I}$ . Then we can construct the space of all valid joining operators  $Q_{ABC}$  as follows. Let  $d_{ABC}$  be the dimension of the composite system. The component along  $A_0 \otimes B_0 \otimes C_0$  is fixed as  $1/d_{ABC}$ , satisfying the trace-one requirement. The components along the two-body operators of the form  $A_i \otimes B_j \otimes \mathbb{I}$  are fixed by the required reduction to  $Q_{AB}$ , and similarly the components along the two-body operators of the form  $A_i \otimes \mathbb{I} \otimes C_k$  are determined by  $Q_{AC}$ . The components along the one-body operators of the form  $A_i \otimes \mathbb{I} \otimes \mathbb{I}$ ,  $\mathbb{I} \otimes B_i \otimes \mathbb{I}$ , and  $\mathbb{I} \otimes \mathbb{I} \otimes C_i$  are determined from

the reductions of  $Q_{AB}$  and  $Q_{AC}$ . This leaves the coefficients of all remaining basis operators unconstrained, since their corresponding basis operators are zero after a partial trace over systems  $B$  or  $C$ .  $\square$

Thus, requiring the joining operator to be Hermitian and normalized is not a limiting constraint with respect to joinability. While this clearly points to the key role played by the non-negativity of the joining operator, our aim here is not to give a general analysis of the manifestation the non-negativity constraint. Rather, what we achieve is a simple characterization of a specific instance of this constraint. A few broader points about non-negativity can nevertheless be made. First, part of the job of non-negativity is to enforce constraints that are also obeyed by classical probability distributions. For example, in the case of a two-qubit state  $\rho$ , if  $\langle X \otimes \mathbb{I} \rangle_\rho = 1$  and  $\langle \mathbb{I} \otimes X \rangle_\rho = 1$ , then  $\langle X \otimes X \rangle_\rho$  must equal 1. More generally, consider a set of mutually commuting observables  $\{M_i\}_{i=1}^k$  and any basis  $\{|m\rangle\}$  in which all  $M_i$  are diagonal. Any valid state must lead to a list of expectation values  $(\text{Tr}(\rho M_1), \dots, \text{Tr}(\rho M_k))$ , whose values are element-wise convex combinations of the vertices  $\{(\langle m|M_1|m\rangle, \dots, \langle m|M_k|m\rangle) | \forall m\}$ . The interpretation of this constraint is that since commuting observables have simultaneously definable values, just as classical observables do, probability distributions on them must obey the rules of classical probability distributions.

Non-negativity constraints that do *not* arise from classical limitations on compatible observables may be labeled as inherently quantum constraints, the most familiar being provided by uncertainty relations for conjugate observables [30, 31]. Although complementarity constraints are most evident for observables acting on the same system, complementarity can also give rise to a trade-off in the information about a subsystem observable vs. a joint observable. This fact is what allows Bell's inequality to be violated. For our purposes, the complementarity that comes into play is that between "overlapping" joint observables (e.g., between  $\vec{S}_1 \cdot \vec{S}_2$  and  $\vec{S}_1 \cdot \vec{S}_3$  for three qubits). We are thus generally interested in understanding the interplay between purely classical and quantum joining limitations, and in the correlation trade-offs that may possibly emerge.

Historically, as already mentioned, a pioneering exploration of the extent to which quantum correlations can be shared among three parties was carried out in [4], yielding a characterization of the monogamy of entanglement in terms of the well-known CKW inequality:

$$\mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 \leq (\mathcal{C}_{A(BC)}^2)^{\min},$$

where  $\mathcal{C}$  denotes the concurrence and the right hand-side is minimized over all pure-state decompositions. Thus, with the entanglement across the bipartition  $A$  and  $(BC)$  held fixed, an increase in the upper bound of the  $A$ - $B$  entanglement can only come at the cost of a decrease in the upper bound of the  $A$ - $C$  entanglement. One may wonder whether the CKW inequality may help in diagnosing

joinability properties of reduced states. If a joining state  $w_{ABC}$  is *not* a priori determined (in fact, the existence of such a state *is* the entire question of joinability), the CKW inequality may in principle be used to obtain a *necessary* condition for joinability, namely, if  $\rho_{AB}$  and  $\rho_{AC}$  are joinable, then

$$\mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 \leq 1. \quad (2)$$

However, there exist pairs of bipartite states – both unentangled (as the following Corollary shows) and non-trivially entangled (as we shall determine in Sec. III.B, see in particular Fig. 3) – that obey the “weak” CKW inequality in Eq. (2), yet are *not* joinable. The key point is that while the limitations that the CKW captures are to be ascribed to entanglement between the subsystems, entanglement is not required to prevent two states from being joinable:

**Corollary II.5.** *Classically correlated quantum states need not be joinable.*

*Proof.* Consider the two quantum states

$$\begin{aligned} \rho_{AB} &= \frac{1}{2}(|\uparrow_X \uparrow_X\rangle\langle\uparrow_X \uparrow_X| + |\downarrow_X \downarrow_X\rangle\langle\downarrow_X \downarrow_X|), \\ \rho_{AC} &= \frac{1}{2}(|\uparrow_Z \uparrow_Z\rangle\langle\uparrow_Z \uparrow_Z| + |\downarrow_Z \downarrow_Z\rangle\langle\downarrow_Z \downarrow_Z|), \end{aligned}$$

on the pairs  $A$ - $B$  and  $A$ - $C$ , respectively. Both have a completely mixed reduced state over  $A$  and thus it is meaningful to consider their joinability. Let  $w_{ABC}$  be a joining state. Then the outcome of Bob's  $X$  measurement would correctly lead him to predict Alice to be in the state  $|\uparrow_X\rangle$  or  $|\downarrow_X\rangle$ , while at the same time the outcome of Charlie's  $Z$  measurement would correctly lead him to predict Alice to be in the state  $|\uparrow_Z\rangle$  or  $|\downarrow_Z\rangle$ . Since this violates the uncertainty principle,  $w_{ABC}$  cannot be a valid joining state.  $\square$

Interestingly, both the above bipartite states are so-called “classical-quantum” states, that is, correlated states which are expressible in the form

$$\rho = \sum_i p_i |i\rangle\langle i|_A \otimes \sigma_B^i, \quad \sum_i p_i = 1,$$

relative to some local orthogonal basis  $\{|i\rangle_A\}$  on  $A$  and with  $\sigma_B^i$  being, for each  $i$ , an arbitrary state on  $B$ . By construction, such (separable) states have vanishing quantum discord [32], showing that quantum correlations beyond entanglement, as captured by this measure, are likewise *not* necessary to prevent joinability in general.

## B. Sharability

As mentioned, the second joinability structure we analyze is motivated by the concept of sharability. In our context, we can think of sharability as a restricted joining



scenario in which a bipartite state is joined with copies of itself. If  $\mathcal{H}^{(2)} \simeq \mathcal{H}_1^{(1)} \otimes \mathcal{H}_2^{(1)}$ , consider a  $N$ -partite space that consists of  $m$  “right” copies of  $\mathcal{H}_1^{(1)}$  and  $n$  “left” copies of  $\mathcal{H}_2^{(1)}$ , with each neighborhood consisting of one right and one left subsystem, respectively (hence a total of  $mn$  neighborhoods). We then have the following:

**Definition II.6. [Quantum Sharability]** A bipartite density operator  $\rho \in \mathcal{D}(\mathcal{H}_L \otimes \mathcal{H}_R)$  is  $m$ - $n$  sharable if there exists an  $N$ -partite density operator  $w \in \mathcal{D}(\mathcal{H}_L^{\otimes m} \otimes \mathcal{H}_R^{\otimes n})$ , called a sharing state, that reduces left-right-pairwise to  $\rho$ , that is,

$$\text{Tr}_{\hat{L}_i \hat{R}_j}(w) = \rho, \quad \forall i = 1, \dots, m, j = 1, \dots, n, \quad (3)$$

where the partial trace is taken over the tensor complement of neighborhood  $ij$ .

Each  $m$ - $n$  sharability scenario may be viewed as a specific joining structure with the additional constraint that each of the joining states be equal to one another:  $\vec{L}_\rho^{m,n} = (\rho, \rho, \dots, \rho)$ . In what follows, we shall take *arbitrarily sharable* to mean  $\infty$ - $\infty$  sharable, whereas *finitely sharable* means that  $\rho$  is not  $m$ - $n$  sharable for some  $m, n$ . Also, each property “ $m$ - $n$  sharable” is referred to as a *sharability criterion*, which a state may or may not satisfy.

It is worth noting the relationship between sharability and the  $N$ -representability problem. The  $N$ -representability problem asks if, for a given (symmetric)  $p$ -partite density operator  $\rho$  on  $(\mathcal{H}_1^{(1)})^{\otimes p}$ , there exists an  $N$ -partite pre-image state for which  $\rho$  is the  $p$ -particle reduced state.  $N$ -representability has been extensively studied for indistinguishable bosonic and fermionic subsystems [12, 13, 33] and is a very important problem in quantum chemistry [34]. We can view  $N$ -representability as a variant on the sharability problem, whereby the distinction between the left and right subsystems is lifted, and  $m + n = p$ . Given the  $p$ -partite state  $\rho$  as the shared state, we ask if there exists a sharing  $N$ -partite state which shares  $\rho$  among all possible  $p$ -partite subsystems. In the setting of indistinguishable particles, the associated symmetry further constrains the space of the valid  $N$ -partite sharing states.

Just as with 1-2 joinability (Proposition II.3), the following result may be established [3]:

**Proposition II.7.** Any classical bipartite probability distribution is arbitrarily sharable.

*Proof.* Let  $p(A, B)$  be the joint probability distribution of two classical random variables  $A$  and  $B$ . By construction, a probability distribution which  $m$ - $n$  shares  $p(A, B)$  is given by

$$w(a_1, \dots, a_m, b_1, \dots, b_n) = \prod_{i=1}^m \prod_{j=1}^n \frac{p(a_i, b_j)}{l(a_i)^n r(b_j)^m}.$$

where  $l(a) = \sum_{b \in B} p(a, b)$  and  $r(b) = \sum_{a \in A} p(a, b)$  denote the marginals of a(ny) left and right random variable, respectively.  $\square$

Similar to the joinability case, convexity properties play an important role towards characterizing sharability. Let  $\dim(\mathcal{H}_1^{(1)}) = d_1 \equiv d_L$  and  $\dim(\mathcal{H}_2^{(1)}) = d_2 \equiv d_R$ . We then have:

**Proposition II.8.** The set of  $m$ - $n$  sharable states form a convex set for given subsystem dimensions  $d_L$  and  $d_R$ .

*Proof.* By Proposition II.2, the set of joinable lists is convex. Thus, since convex combination of joining-lists preserves the equality of list members, the set of  $m$ - $n$  sharable states is convex for fixed subsystem dimensions  $d_L$  and  $d_R$ .  $\square$

The above proposition implies that if  $\rho$  satisfies a particular sharability criterion, then any mixture of  $\rho$  with the completely mixed state also satisfies that criterion, since the completely mixed state is arbitrarily ( $\infty$ - $\infty$ ) sharable. Besides mixing with the identity, the degree of sharability may be unchanged under more general transformations on the input state. Consider, specifically, completely-positive trace-preserving bipartite maps  $\mathcal{M}(\rho)$  that can be written as mixture of local unitary operations, that is,

$$\mathcal{M}(\rho) = \sum_i \lambda_i U_1^i \otimes V_2^i \rho U_1^{i\dagger} \otimes V_2^{i\dagger}, \quad \sum_i \lambda_i = 1, \quad (4)$$

where  $U_1^i$  and  $V_2^i$  are arbitrary unitary transformations  $\mathcal{H}_L$  and  $\mathcal{H}_R$ , respectively. Clearly, maps of the above forms form a proper (unital) subset of general Local Operations and Classical Communication (LOCC) [1]. We can thus establish the following:

**Theorem II.9.** If  $\rho$  is  $m$ - $n$  sharable, then  $\mathcal{M}(\rho)$  is  $m$ - $n$  sharable for any unital LOCC map  $\mathcal{M}$ .

*Proof.* Let  $\mathcal{M}(\rho)$  be expressed as in Eq. (4). By virtue of Proposition II.8, it suffices to show that each term in the sum is  $m$ - $n$  sharable, that is, that if a state  $\rho$  is  $m$ - $n$  sharable, then a state  $\rho_{UV}$  obtained from  $\rho$  via a local unitary transformation  $U \otimes V$  is also  $m$ - $n$  sharable. Let  $w$  be a sharing state for  $\rho$ , and define

$$w' = (U_1 \dots U_m V_{m+1} \dots V_{m+n}) w (U_1^\dagger \dots U_m^\dagger V_{m+1}^\dagger \dots V_{m+n}^\dagger).$$

Then, for any left-right pair of subsystems  $i$  and  $j$ , it follows that

$$\text{Tr}_{i,j}(w') = U_i V_j \text{Tr}_{i,j}(w) U_i^\dagger V_j^\dagger = U \otimes V \rho U^\dagger \otimes V^\dagger = \rho_{UV}.$$

Hence,  $w'$  is an  $m$ - $n$ -sharing state for  $\rho_{UV}$ , as desired.  $\square$

This result suggests a connection between the degree of sharability and the entanglement of a given state. In both cases, there exist classes of states for which these properties cannot be “further degraded” by LOCC. Specifically, LOCC cannot decrease the entanglement of states with no entanglement and cannot increase the sharability of states with  $\infty$ - $\infty$  sharability. These two classes of states can in fact be shown to coincide, as a consequence of the fact that *arbitrary sharability is equivalent to (bipartite)*

*separability.* While such a connection relies crucially on a theorem on symmetric extensions established in [35] and has been appreciated in the literature [2, 3, 36], we explicitly address it here in view of its significance:

**Theorem II.10.** *A bipartite quantum state  $\rho$  on  $\mathcal{H}_L \otimes \mathcal{H}_R$  is unentangled (or separable) if and only if it is arbitrarily sharable.*

*Proof.* ( $\Leftarrow$ ) Let  $\rho$  be separable. Then for some set of density operators  $\{\rho_i^L, \rho_i^R\}$ , it can be written as

$$\rho = \sum_i \lambda_i \rho_i^L \otimes \rho_i^R,$$

with  $\sum_i \lambda_i = 1$ . Let  $n$  and  $m$  be arbitrary, and let the  $N$ -partite state  $w$ , be defined as follows:

$$w = \sum_i \lambda_i (\rho_i^L)^{\otimes m} \otimes (\rho_i^R)^{\otimes n},$$

with  $N = m + n$ . By construction, the state of each  $L$ - $R$  pair is  $\rho$ , since it follows straightforwardly that Eq. (3) is obeyed for each  $i, j$ . Thus,  $w$  is a valid sharing state.

( $\Rightarrow$ ) Since  $\rho$  is arbitrarily sharable, there exists a sharing state  $w$  for arbitrary values of  $m, n$ . In particular, we need only make use of a sharing state  $w$  for  $m = 1$  and arbitrarily large  $n$ , whence we let  $n \rightarrow \infty$ . Given  $w$ , let us construct another sharing state  $\tilde{w}$ , which is invariant under permutations of the right subsystems, that is, let

$$\tilde{w} = \frac{1}{|S_R|} \sum_{\pi \in S_R} \pi(w),$$

where  $S_R \equiv \{\pi\}$  is the group of the permutations operators on  $\mathcal{H}_R^{\otimes n}$ , with the action of a permutation on pure and mixed states being given by  $\pi|\psi_1\rangle|\psi_2\rangle\ldots|\psi_n\rangle = |\psi_{\pi(1)}\rangle|\psi_{\pi(2)}\rangle\ldots|\psi_{\pi(n)}\rangle$  and  $\pi(w) = \pi^\dagger w \pi$ , respectively. It then follows that  $\tilde{w}$  shares  $\rho$ :

$$\begin{aligned} \text{Tr}_{\hat{L}, \hat{R}}(\tilde{w}) &= \frac{1}{|S_R|} \sum_{S_R} \text{Tr}_{\hat{L}, \hat{R}}(\pi(w)) \\ &= \frac{1}{|S_R|} \sum_{S_R} \text{Tr}_{\hat{L}, \pi(\hat{R}_i)}(w) \\ &= \frac{1}{|S_R|} \sum_{S_R} \rho = \rho. \end{aligned}$$

Having established the existence of a symmetric sharing state  $\tilde{w} \in \mathcal{D}(\mathcal{H}_L \otimes \mathcal{H}_R^{\otimes \infty})$ , Fannes' Theorem (see section 2 of [35]) implies the existence of a unique representation of  $\tilde{w}$  as a sum of product states,

$$\tilde{w} = \sum_i \lambda_i \rho_L^i \otimes \rho_R^i \otimes \rho_R^i \otimes \ldots$$

Reducing  $\tilde{w}$  to any  $L$ - $R$  pair leaves a separable state. Thus, if  $\rho$  is 1- $n$  sharable it must be separable.  $\square$

As we alluded to before, a Corollary of this result is that in fact 1- $\infty$  sharability implies  $\infty$ - $\infty$  sharability. In closing this section, we also briefly mention the concept of *exchangeability* [37]. A density operator  $\rho$  on  $(\mathcal{H}_1^{(1)})^{\otimes p}$  is said to be exchangeable if it is symmetric under permutation of its  $p$  subsystems and if there exists a symmetric state  $w$  on  $(\mathcal{H}_1^{(1)})^{\otimes (p+q)}$  such that the reduced states of any subset of  $p$  subsystems is  $\rho$  for all  $q \in \mathbb{N}$ . Similar to sharability, exchangeability implies separability. However, the converse only holds in general for sharability: clearly, there exist states which are separable but not exchangeable, because of the extra symmetry requirement. Thus, the notion of sharability is more directly related to entanglement than exchangeability is.

### III. JOINING AND SHARING WERNER STATES

Even for the simplest case of two bipartite states with an overlapping marginal, a general characterization of joinability is extremely non-trivial. To the best of our knowledge, no conditions exist which are both necessary and sufficient for two generic density operators to be joinable; although, conditions that are separately necessary or sufficient have been recently derived [23]. To make headway in this problem, we obtain a complete characterization for the three-party joining scenario and for sharability at the cost of narrowing the scope of the states we consider to Werner states.

The utility of bipartite Werner states is derived from their simple analytic properties and range of mixed state entanglement. For a given subsystem dimension  $d$ , Werner states are defined as the one-parameter family which are invariant under collective unitary transformations [24], that is, transformations of the form  $U \otimes U$ , for arbitrary  $U \in \mathfrak{U}(d)$ . The parameterization which we employ is

$$\rho(\Phi) = \frac{1}{d^3 - d} [(d - \Phi)\mathbb{I} + (d\Phi - 1)V], \quad (5)$$

where  $V$  is the swap operator, defined by its action on any product ket,  $V|\psi\phi\rangle \equiv |\phi\psi\rangle$ , and non-negativity is ensured by  $-1 \leq \Phi \leq 1$ . By construction, the completely mixed state corresponds to  $\Phi = 1/d$ . A nice feature of the parameterization in Eq. (5) is the property  $\Phi = \text{Tr}(V\rho(\Phi))$ . Furthermore, the concurrence of Werner states is simply given by [38]

$$\mathcal{C}(\rho(\Phi)) = -\text{Tr}[V\rho(\Phi)] = -\Phi, \quad \Phi < 0. \quad (6)$$

For  $\Phi \geq 0$ , the concurrence is defined to be zero, indicating separability. Werner states have been experimentally characterized for photonic qubits, see e.g. [39]. Interestingly, they can be dissipatively prepared as the steady state of coherently driven atoms subject to collective spontaneous decay [40].

We present another way to think of Werner states, which will prove useful later. First, the purity of the

$\Phi = -1$  state is  $2/d(d-1)$ . For  $d = 2$  this purity is 1, corresponding to the pure singlet state. But, for  $d > 2$  this purity is less than 1. Second, collective projective measurements on a most-entangled Werner state return disagreeing outcomes (e.g., corresponding to  $|1\rangle$  and  $|3\rangle$ , but not  $|1\rangle$  and  $|1\rangle$ ). The following construction of bipartite Werner states demonstrates the origin of both of these essential features. The  $d > 2$  analogue to the singlet state is the following  $d$ -partite anti-symmetric state:

$$|\psi_d^-\rangle = \sum_{\pi \in S_d} \text{sign}(\pi) V_\pi |1\rangle |2\rangle \dots |d\rangle, \quad (7)$$

where as before  $S_d \equiv \{\pi\}$  denotes the permutation group and  $\{|\ell\rangle\}$  is an orthonormal basis on  $\mathcal{H}^{(1)} \simeq \mathbb{C}^d$ . This state has the property that a collective measurement will return outcomes that differ on each qudit with certainty. The most-entangled *bipartite* qudit Werner state is the two-party reduced state of  $|\psi_d^-\rangle$ . Thus, we can think of bipartite qudit Werner states as mixtures of the completely mixed state with the two-party-reduced “completely disagreeing” state  $|\psi_d^-\rangle$ . The inverse of  $2/d(d-1)$  (the purity) is precisely the number of terms in Eq. (7), which is the number of ways two “dits” can disagree. Understanding bipartite Werner states to arise from reduced states of  $|\psi_d^-\rangle$  will help us understand some of the results of Sec. IIIB and IIIC.

Since each subsystem in a Werner state is completely mixed, the probability of obtaining any outcome from a projective measurement on a single system is  $1/d$ . Hence,  $\Phi$  has no bearing on subsystem-measurement outcomes. Instead,  $\Phi$  is directly related to the *correlation* between measurement outcomes. Consider a projective measurement on each subsystem. Let  $|\psi\rangle$  be an outcome state of the measurement on  $A$  and, similarly,  $|\phi\rangle$  for  $B$ . The conditional probability of obtaining  $|\phi\rangle$  given  $|\psi\rangle$  is

$$\begin{aligned} p(|\phi\rangle_B | |\psi\rangle_A) &= \frac{p(|\phi\rangle_B, |\psi\rangle_A)}{p(|\psi\rangle_A)} \\ &= \frac{|\langle\psi|\rho(\Phi)|\psi\rangle|^2}{|\langle\psi|\text{Tr}_2[\rho(\Phi)]|\psi\rangle|^2} \\ &= \frac{1}{d} + \frac{d(\Phi - 1/d)(|\langle\psi|\phi\rangle|^2 - 1/d) - 1}{d^2 - 1}. \end{aligned}$$

In the case that  $|\phi\rangle = |\psi\rangle$ , this expression reduces to

$$p(B = |\psi\rangle | A = |\psi\rangle) = \frac{\Phi + 1}{d + 1} \equiv \alpha. \quad (8)$$

Thus, the probability of agreement is independent of the choice of  $|\psi\rangle$ , and depends only upon  $\Phi$ . In other words, we may speak unambiguously about the *probability of agreement*,  $\alpha$ , of a Werner state. Re-parameterizing these states in terms of  $\alpha$ , we obtain

$$\rho(\alpha) = \left(1 - \frac{d\alpha - 1}{d - 1}\right) \mathbb{I} + \frac{d\alpha - 1}{d - 1} V. \quad (9)$$

The parametrization in Eq. (8) allows a classical analogue to the Werner state family to be naturally introduced. Specifically, consider a probability distribution on

the outcome space  $\Omega_d \times \Omega_d = \{1, \dots, d\} \times \{1, \dots, d\}$ , for which the marginal distributions are completely mixed and for which the overall distribution is symmetric about a swap of the two systems. The classical two-party Werner states interpolate between an even mixture of “agreeing pure states”, namely,  $(1, 1), (2, 2), \dots, (d, d)$ , and an even mixture of all possible “disagreeing pure states”, namely,  $(1, 2), \dots, (1, d), (2, 1), \dots, (d, d-1)$ . The resulting probability distribution may be written as

$$p(A = i, B = j)_\alpha = \frac{\alpha}{d} \delta_{i,j} + \frac{1 - \alpha}{d(d-1)} (1 - \delta_{i,j}). \quad (10)$$

### A. Joinability of classical Werner states

In order to determine whether joinability limitations exist in the classical case, we begin by noting that any (finite-dimensional) classical probability distribution is a unique convex combination of the pure states of the system. In our case, there are five extremal three-party states, for which the two-party marginals are classical Werner states, as defined in Eq. (10). These are

$$\begin{aligned} p(A, B, C \text{ agree}) &= \frac{1}{d} \sum_i (i, i, i), \\ p(A, B \text{ agree}) &= \frac{1}{d(d-1)} \sum_{i \neq j} (i, i, j), \\ p(A, C \text{ agree}) &= \frac{1}{d(d-1)} \sum_{i \neq j} (i, j, i), \\ p(B, C \text{ agree}) &= \frac{1}{d(d-1)} \sum_{i \neq j} (j, i, i), \\ p(\text{all disagree}) &= \frac{1}{d(d-1)(d-2)} \sum_{i \neq j \neq k} (i, j, k), \end{aligned}$$

where  $(i, j, k)$  stands for the pure distribution  $p(A, B, C) = \delta_{A,i} \delta_{B,j} \delta_{C,k}$ . The first four of these states are valid for all  $d \geq 2$  and each correspond to a vertex of a tetrahedron, as depicted in Fig. 1. The fifth state is only valid for  $d \geq 3$  and corresponds to the point  $(\alpha_{AB}, \alpha_{AC}, \alpha_{BC}) = (0, 0, 0)$ . Any valid three-party state for which the two-party marginals are classical Werner states must be a convex combination of the above states. Therefore, the joinable-unjoinable boundary is delimited by the boundary of their convex hull. For the  $d = 2$  case, the inequalities describing these boundaries are explicitly given by the following:

$$\begin{aligned} p(A, B, C \text{ agree}) \geq 0 &\Rightarrow \alpha_{AB} + \alpha_{AC} + \alpha_{BC} \geq 1, \\ p(C \text{ disagrees}) \geq 0 &\Rightarrow \alpha_{AB} - \alpha_{AC} - \alpha_{BC} \geq 1, \\ p(B \text{ disagrees}) \geq 0 &\Rightarrow -\alpha_{AB} + \alpha_{AC} - \alpha_{BC} \geq 1, \\ p(A \text{ disagrees}) \geq 0 &\Rightarrow -\alpha_{AB} - \alpha_{AC} + \alpha_{BC} \geq 1, \end{aligned}$$

where each inequality arises from requiring that the corresponding extremal state has a non-negative likelihood.

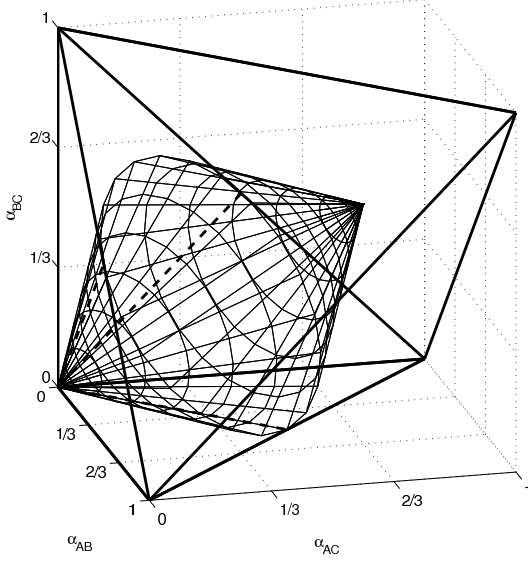


FIG. 1. Quantum and classical joinability limitations for three pairs of Werner states, as parametrized in Eqs. (5) and (10), respectively. The quantum boundary is the surface of the bicone with its vertices at  $(2/3, 2/3, 2/3)$  and  $(0, 0, 0)$ , while the classical boundary is the surface of the two joined tetrahedra. In the qubit and bit cases ( $d = 2$ ), the back face of the larger tetrahedron delineates a joinability boundary and hence all points in the lower cone and smaller tetrahedron, respectively, are off limits.

### B. Joinability of qudit Werner states

We now proceed to determine which pairs of bipartite Werner states  $\rho_{AB}$  and  $\rho_{AC}$  can be joined. Our starting point is the fact that if a three-partite state  $w_{ABC}$  joins Werner states  $\rho_{AB}$  and  $\rho_{AC}$ , then the “twirled state”  $\tilde{w}_{ABC}$ , given by

$$\tilde{w}_{ABC} = \int (U \otimes U \otimes U) w_{ABC} (U \otimes U \otimes U)^\dagger d\mu(U), \quad (11)$$

is also a valid joining state. In Eq. (11),  $\mu$  denotes the invariant Haar measure on  $\mathfrak{U}(d)$  and the twirling super-operator effects a projection into the subspace of operators with collective unitary invariance [41]. By invoking the Shur-Weyl duality [42], the guaranteed existence of joining states with this symmetry allows us to narrow the search for tripartite joining states to the Hermitian subspace spanned by representations of subsystem permutations, that is, density operators of the form

$$w = \sum_{\pi \in S_3} \mu_\pi V_\pi, \quad (12)$$

where Hermiticity demands that  $\mu_\pi^* V_\pi^\dagger = \mu_\pi V_\pi$ .

To determine which pairs of Werner states  $\rho_{AB}$  and  $\rho_{AC}$  are joinable, we will find it easier to first tackle the more symmetric problem of determining which trios of Werner states  $\rho_{AB}$ ,  $\rho_{AC}$  and  $\rho_{BC}$  are joinable. Given  $w_{ABC}$  which joins Werner states, each subsystem pair is

characterized by the expectation value with the respective swap operator,

$$\Phi_{ij} = \text{Tr} \left( w_{ABC} V_{ij} \otimes \mathbb{I}_{ij} \right), \quad (13)$$

where  $i, j \in \{A, B, C\}$  with  $i \neq j$ . Hence, we seek to determine for which  $(\Phi_{AB}, \Phi_{BC}, \Phi_{AC})$  there exists a density operator  $w_{ABC}$  which fulfills the prescription of Eq. (13). Our main results is the following:

**Theorem III.1.** *Three bipartite qudit Werner states with parameters  $\Phi_{AB}, \Phi_{BC}, \Phi_{AC}$  are joinable if  $(\Phi_{AB}, \Phi_{BC}, \Phi_{AC})$  lies within the bicone described by*

$$1 \pm \bar{\Phi} \geq \frac{2}{3} |\Phi_{BC} + \omega \Phi_{AC} + \omega^2 \Phi_{AB}|, \quad (14)$$

for  $d \geq 3$ , or within the cone described by

$$1 - \bar{\Phi} \geq \frac{2}{3} |\Phi_{BC} + \omega \Phi_{AC} + \omega^2 \Phi_{AB}|, \quad \bar{\Phi} \geq 0, \quad (15)$$

for  $d = 2$ , where

$$\bar{\Phi} = \frac{1}{3} (\Phi_{AB} + \Phi_{BC} + \Phi_{AC}), \quad \omega = e^{i\frac{2\pi}{3}}. \quad (16)$$

*Proof.* Following [41], we reparameterize the space described by Eq. (12) in terms of operators whose actions are restricted to irreducible subspaces. This allows us to enforce non-negativity in a straightforward manner. Specifically (see Appendix A), the relevant parameters are the expectation values  $r_k = \text{Tr}(w R_k)$  of the operators defined in Eqs. (A2) and (A3), and non-negativity is enforced by  $r_+, r_-, r_0 \geq 0$  and the spherical inequality  $r_1^2 + r_2^2 + r_3^2 \leq r_0^2$ .

The joinability limitations are obtained by expressing the above non-negativity constraints in terms of the Werner parameters  $\Phi_{ij}$ . This is done by explicitly relating the  $r_k$  to the  $\Phi_{ij}$ ,

$$\begin{aligned} r_1^2 + r_2^2 &= \text{Tr}(w R_1)^2 + \text{Tr}(w R_2)^2 \\ &= |\text{Tr}(w(R_1 + i R_2))|^2 \\ &= \frac{4}{9} \left| \text{Tr} \left[ w \left( V_{(BC)} + e^{i\frac{2\pi}{3}} V_{(CA)} + e^{i\frac{4\pi}{3}} V_{(AB)} \right) \right] \right|^2 \\ &= \frac{4}{9} |\Phi_{BC} + \omega \Phi_{AC} + \omega^2 \Phi_{AB}|^2, \end{aligned}$$

and

$$\begin{aligned} r_0 &= \text{Tr}(w(R_0 + 2R_-)) - 2\text{Tr}(w R_-) \\ &= \text{Tr} \left[ w \left( \mathbb{I} - \frac{1}{3} (V_{(AB)} + V_{(BC)} + V_{(CA)}) \right) \right] - 2r_- \\ &= \frac{1}{3} \sum_{i < j} [(1 - 2r_-) - \Phi_{ij}] \\ &= 1 - 2r_- - \bar{\Phi}, \end{aligned} \quad (17)$$

where  $\bar{\Phi}$  is defined in Eq. (16). Thus, the spherical inequality may be rewritten as

$$(1 - 2r_- - \bar{\Phi})^2 \geq \frac{4}{9} |\Phi_{BC} + \omega \Phi_{AC} + \omega^2 \Phi_{AB}|^2 + r_3^2. \quad (18)$$



Since a non-zero value of  $r_3$  only further limits the inequality and since the  $\Phi_{ij}$  are independent of it, we maximize the range of joinable  $\Phi_{ij}$  by setting  $r_3 = 0$ .

The non-negativity is then expressed in terms of the  $\Phi_{ij}$  and  $r_-$  as

$$1 - 2r_- - \bar{\Phi} \geq \frac{2}{3} |\Phi_{BC} + \omega\Phi_{AC} + \omega^2\Phi_{AB}|, \quad (19)$$

$$\bar{\Phi} + r_- \geq 0, \quad (20)$$

$$r_- \geq 0, \quad (21)$$

where the normalization condition allows us to write the  $r_+$ -non-negativity condition as Eq. (20) and the non-negativity of  $r_0$  allows us to take the square-root of Eq. (18) to obtain Eq. (19).

For each  $\bar{\Phi}$ , we set  $r_-$  so as to maximize the left hand side of Eq. (19) while satisfying Eq. (20) and Eq. (21). Let  $d \geq 3$ . For  $\bar{\Phi} \geq 0$  we set  $r_- = 0$ , while for  $\bar{\Phi} \leq 0$ , we set  $r_- = -\bar{\Phi}$ . Considering these two cases together, we find that the region of joinable  $(\Phi_{AB}, \Phi_{BC}, \Phi_{AC})$  is given precisely by Eq. (14). If  $d = 2$ , we have  $r_- = 0$ , thus simplifying Eq. (19) and Eq. (20) to Eq. (15), as desired.  $\square$

The results of Theorem III.1 are pictorially summarized in Fig. 2. Having determined the joinable trios of Werner states, we may now answer the question of what  $A$ - $B$  and  $A$ - $C$  Werner states are joinable with one another:

**Corollary III.2.** *Two pairs of qudit Werner states with parameters  $\Phi_{AB}$  and  $\Phi_{AC}$  are joinable if and only if  $\Phi_{AB}, \Phi_{AC} \geq -\frac{1}{2}$ , or if the parameters satisfy*

$$(\Phi_{AB} + \Phi_{AC})^2 + \frac{1}{3}(\Phi_{AB} - \Phi_{AC})^2 \leq 1, \quad (22)$$

or additionally, in the case  $d \geq 3$ , if  $\Phi_{AB}, \Phi_{AC} \leq \frac{1}{2}$ .

*Proof.* To obtain these conditions, it suffices to project the shape given Eq. (15) onto the  $\Phi_{AB}$ - $\Phi_{AC}$  plane. The rim of the cone/bicone projects down to an ellipse whose equation we obtain by extremizing Eq. (14) evaluated at the cone base  $\bar{\Phi} = 0$ . Setting  $\Phi_{BC} = -\Phi_{AB} - \Phi_{AC}$ , we find the boundary to be  $1 = (\Phi_{AB} + \Phi_{AC})^2 + \frac{1}{3}(\Phi_{AB} - \Phi_{AC})^2$ . Any pairs of Werner states within this ellipse are joinable. We also have that  $\Phi_{AB} = 1$  and  $\Phi_{AC} = 1$  are joinable since setting  $\Phi_{BC} = 1$  causes the three to satisfy Eq. (14). Then, by Proposition II.2, the convex hull of  $(\Phi_{AB}, \Phi_{AC}) = (1, 1)$  and the ellipse corresponds to pairs of joinable states. If  $d \geq 3$ , the states  $\Phi_{AB} = -1$  and  $\Phi_{AC} = -1$  are joinable by setting  $\Phi_{BC} = -1$ , and hence the joinable  $(\Phi_{AB}, \Phi_{AC})$  pairs also include the convex hull of the ellipse with the point  $(\Phi_{AB}, \Phi_{AC}) = (-1, -1)$ .

It remains to show that if  $\Phi_{AB}$  or  $\Phi_{AC} \leq -\frac{1}{2}$  (additionally,  $\Phi_{AB}$  or  $\Phi_{AC} \geq \frac{1}{2}$  for  $d \geq 3$ ), then  $(\Phi_{AB}, \Phi_{AC})$  pairs outside of the ellipse are not joinable. To achieve this, we consider a cone viewed from an arbitrary direction an infinite distance away. The shape seen is the shape of the

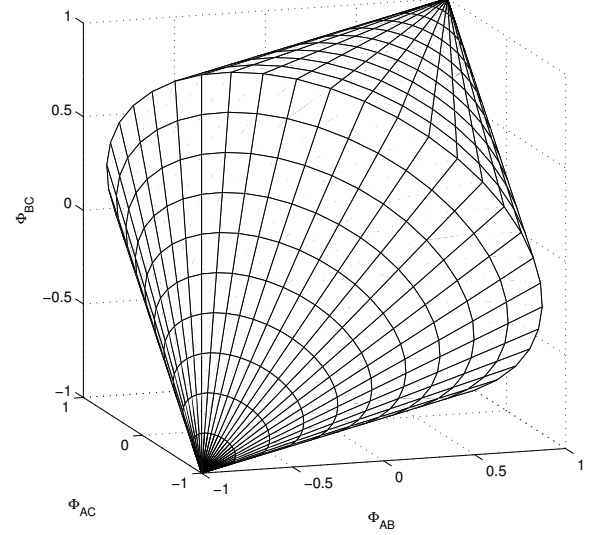


FIG. 2. The bicone containing the joinable trios of qudit Werner states. For  $d = 2$ , the joinable region is restricted to just the cone with vertex at  $(1, 1, 1)$ .

projection. From this vantage point, the circular base of the cone appears as an ellipse. The remaining visible area (seen only if the vertex does not overlap with the base) constitutes the projection of the cone's lateral surface. The boundary of this projection is defined by the two lines extending from the vertex that are tangent to the ellipse. The area contained between these two lines along with the hull of the ellipse constitutes the shape visible from the infinity perspective, or, in other words, the cone's projection. In our case, the points at which these two lines (four lines for the bicone) intersect the ellipse are  $(-1/2, 1)$  and  $(1, -1/2)$  (additionally,  $(-1, 1/2)$  and  $(1/2, -1)$  for the bicone). Beyond these points, the ellipse "takes over" as the projection boundary delimiter. Thus, points satisfying  $\Phi_{AB}$  or  $\Phi_{AC} \leq -\frac{1}{2}$  (and,  $\Phi_{AB}$  or  $\Phi_{AC} \geq \frac{1}{2}$  for the bicone) are joinable if and only if they are within the ellipse boundary.  $\square$

The above result gives the exact quantum mechanical rule for the two-pair joinability of Werner states, as pictorially summarized in see Fig. 3. A number of interesting features are worth noticing. First, by restricting to the line where  $\Phi_{AB} = \Phi_{AC}$ , we can conclude that qubit Werner states are 1-2 sharable if and only if  $\Phi \geq -\frac{1}{2}$ , whereas for  $d \geq 3$ , all qudit Werner states are 1-2 sharable. As we shall see, this agrees with the more general analysis of Sec. III C.

Second, some insight into the role of entanglement in limiting joinability may be gained. In the first quadrant of Fig. 3, where neither pair is entangled, it is no surprise that no joinability restrictions apply. Likewise, it is not surprising to see that, in the third quadrant where both pairs are entangled, there is a trade-off between the

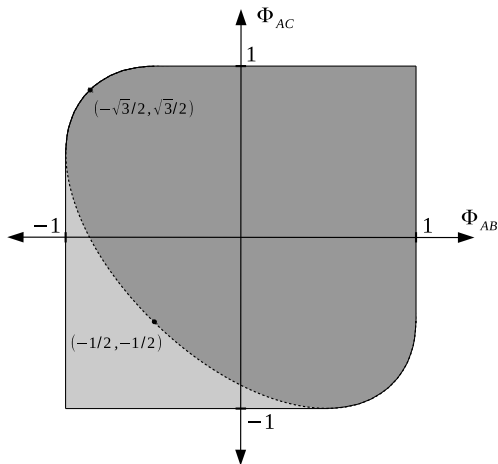


FIG. 3. (Color online) The shaded region corresponds to joinable Werner pairs, with the lighter region being valid only for  $d \geq 3$ . The rounded (dashed) boundary is the ellipse determined by Eq. (22). This explicitly shows the existence of pairs of entangled Werner states that are within the circular boundary determined by the weak CKW inequality, Eq. (2), yet are not joinable.

amount of entanglement allowed between one pair and that of the other. But, in the second and fourth quadrants we observe a more interesting behavior. Namely, these quadrants show that there is also a *trade-off* between the amount of classical correlation in one pair and the amount of entanglement in the other pair. In fact, the smoothness of the boundary curve as it crosses from one of the pairs being entangled to unentangled suggests that, at least in this case, entanglement is not the correct figure of merit in diagnosing joinability limitations.

We can further compare these quantum joinability limitations to the joinability limitations in place for classical Werner states. As described in Sec. III A, the non-negativity of  $p(A, B, C \text{ agree})$  is enforced by the inequality  $\alpha_{AB} + \alpha_{AC} + \alpha_{BC} \geq 1$ . We expect the same requirement to be enforced by the analogue quantum-measurement statistics. The base of the qubit joinability-limitation cone is determined by  $\Phi_{AB} + \Phi_{AC} + \Phi_{BC} \geq 0$ . Writing each of the Werner parameters in terms of the probability of agreement  $\alpha$  defined in Eq. (8), we obtain  $\alpha_{AB} + \alpha_{AC} + \alpha_{BC} \geq 1$ . Hence, part of the quantum joining limitations are indeed derived from the classical joining limitations. This is also illustrated in Fig. 1. Of course, one would not expect the quantum scenario to exhibit violations of the classical joinability restrictions, still, it is interesting that states which exhibit manifestly non-classical correlations may nonetheless saturate bounds obtained from purely classical joining limitations.

Note that in the case of joining two Werner pairs, entanglement in at least one of the pairs is *necessary* for a limitation on joining. However, when considering the  $A$ - $B$ - $C$  three-pair joining scenario, there exist *trios* of unentangled Werner states which are not joinable. For

example, the point

$$(\Phi_{AB}, \Phi_{AC}, \Phi_{BC}) = (1, 1, 0)$$

corresponds to three separable Werner states that are not joinable. This point is of particular interest because its classical analogue *is* joinable. Translating  $(1, 1, 0)$  into the agreement-probability coordinates,  $(\alpha_{AB}, \alpha_{AC}, \alpha_{BC}) = (2/3, 2/3, 1/3)$ , we see that this point is actually on the classical joining limitation border. Thus, these three separable, correlated states are not joinable for purely quantum mechanical reasons.

Before moving on to sharability, we describe how the above analysis can be generalized in principle. We have thus far considered joining three bipartite Werner states in a triangular fashion. For a  $N$ -partite system, one may want to answer the following question: Which sets of  $N(N-1)/2$  Werner-state pairs are joinable? The approach to solving this more general problem parallels the specific three-party case.

If each pair is in a Werner state, then if a joining state exists, there must exist a joining state with collective invariant symmetry (that is, invariant under arbitrary collective unitaries  $U^{\otimes N}$ ). Thus, we need only look in the set of states with this property. Any operator on  $N$  systems with this symmetry may be decomposed into a sum of operators which each have support on just a single irreducible subspace. This is useful because positivity of the joining operator when restricted to each irreducible subspace is sufficient for positivity of the overall operator. The joining operators may then be decomposed into the projectors on each irreducible subspace and corresponding bases of traceless operators on the projectors. The basis elements will be combinations of permutation operators and the dimension of each such operator subspaces is given by the square of the hook length of the corresponding Young diagram [43]. The remaining task is to obtain a characterization of the positivity of the operators on each subspace. In [44], for example, a method for characterizing the positivity of low-dimensional operator spaces is presented. As long as the number of subsystems remains small, this approach grants us a more computationally friendly characterization of positivity of the joining states. The bounds on the joinable Werner pairs may then be obtained by projecting the positivity characterization boundary onto the space of Werner pairs, analogous to the space of Fig. 3.

### C. Sharability of Werner states

We begin by obtaining a formula that classifies the set of  $1$ - $n$  sharable Werner states. We then outline a procedure for determining the more general  $m$ - $n$  sharability of Werner states and describe how this procedure can be generalized to determine which Werner states satisfy more general sharability structures, akin to the *entanglement molecules* of Dür [15] (see also [45]). Our main result is contained in the following:

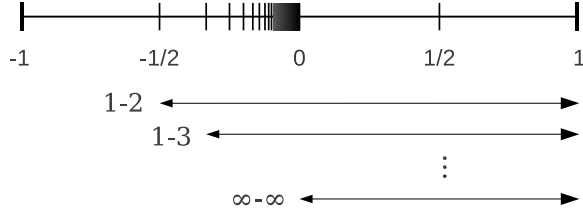


FIG. 4. Pictorial summary of sharability properties of qubit Werner states, according to Eq. (23). The arrow-headed lines depict the parameter range for which Werner states satisfy each of the sharability properties displayed to the left. The vertical ticks between  $\Phi = -1/2$  and 0 mark the left end points of these ranges. These are the points at which subsequent 1- $n$  sharability properties begin to be satisfied. Recall that with this parametrization,  $\Phi = -1$  corresponds to the singlet state whilst  $\Phi = 1/2$  to the fully mixed one. With the exception of the latter, all Werner states exhibit non-vanishing quantum discord [19].

**Theorem III.3.** *A qudit Werner state with parameter  $\Phi$  is 1- $n$  sharable if and only if*

$$\Phi \geq -\frac{d-1}{n}. \quad (23)$$

A pictorial representation of the above result, specialized to qubits, is given in Fig. 4. Interestingly, Eq. (23) implies that only for the qubit case does a *finite* parameter range exist, where the corresponding Werner states are *not* sharable. For  $d \geq 3$ , each Werner state is at least 1-2 sharable. This simply reflects the fact that  $|\psi_d^-\rangle$  (recall Eq. (7)) provides a 1- $(d-1)$  sharing state for a most-entangled qudit Werner state. We now proceed to a formal proof of the above Theorem:

*Proof.* For later reference, we shall set up the necessary mathematical framework for the general  $m$ - $n$  sharability scenario, and then draw upon the representation-theoretic tools of Appendix B to obtain the specific 1- $n$  sharability result. For fixed  $m, n$ , we only need to determine the most-entangled Werner state (largest value of  $-\Phi$ ) satisfying that property; as noted in Sec. II B, all mixtures of that state with any separable state will necessarily satisfy the sharability property, thus in the one-dimensional convex set parameterized by  $\Phi$ , the most-entangled Werner state that satisfies a sharability property indicates the boundary between the satisfying and the failing region.

The next step is to map the problem of determining the most-entangled Werner state for a given sharability criterion to the problem of determining the maximal eigenvalue of a particular operator. The concurrence of a Werner state is measured by its expectation value with respect to  $-V$  [Eq. (6)]. Thus, given a composite system state  $|\psi\rangle$ , if a bipartite subsystem  $A$ - $B$  is in a Werner state, then its concurrence is  $-\Phi_{AB} = -\langle\psi|V_{AB}\otimes\mathbb{I}_{\overline{AB}}|\psi\rangle$ . For each sharability criterion, we then seek an operator (the sharing state) which has maximal expectation value

with respect to a sum of negative swap operators, namely,

$$H_{m,n} = \frac{1}{mn} \sum_{i \in L} \sum_{j \in R} (-V_{ij}),$$

and which satisfies left-system permutation invariance, right-system permutation invariance, and collective unitary invariance. It follows that the eigenspaces of  $H_{m,n}$  are preserved under these symmetry transformations as well. Thus, a suitable sharing state is given by the normalized projector into the maximal-eigenvalue eigenspace of  $H_{m,n}$ , which we denote  $w_{m,n}$ . Explicitly, we have

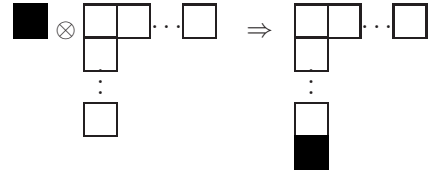
$$\begin{aligned} \text{Tr}(H_{m,n}w_{m,n}) &= \frac{1}{mn} \sum_{i \in L} \sum_{j \in R} -\text{Tr}(V_{ij} \text{Tr}_{\overline{ij}}(w_{m,n})) \\ &= \frac{1}{mn} \sum_{i \in L} \sum_{j \in R} (-\Phi_{ij}) \\ &= -\Phi, \end{aligned}$$

where in the second line we have used the collective unitary invariance of  $w_{m,n}$  to replace each  $V_{ij}$  expectation value with the Werner parameter of its  $i$ - $j$  reduced state, and in the third line we have used the left- and right-subsystem permutation invariance to equate each  $\Phi_{ij}$ .

Thus, the maximal eigenvalue of  $H_{m,n}$  is the concurrence of the most-entangled Werner state which is  $m$ - $n$  sharable, or, in other words, no Werner state with a larger concurrence can be  $m$ - $n$  sharable. It follows that, by determining the largest eigenvalue of each  $H_{m,n}$ , we will have characterized the sharability properties of all Werner states. A detailed calculation of these eigenvalues is found in Appendix B. Here, we outline the main steps and state the results. From Eq. (B1), the following exact expression is found for the 1- $n$  sharing case:

$$H_{1,n} = \frac{1}{2n} (\Lambda_{L_1}^2 + \Lambda_{R_n}^2 - \Lambda_{L_1 R_n}^2) - \frac{\mathbb{I}}{d}.$$

The eigenvalue of  $H_{1,n}$  is maximized by the following gluing of the 1 and  $n$ -box Young diagrams



The values of the  $A$  and  $B$  here are  $A_{Y_{LR}} = d^2 + n + 1 - d$ ,  $A_{Y_L} = 1$ ,  $A_{Y_R} = (d-1)^2 + n + 1 - d$ , and  $B_{Y_{LR}} = B_{Y_L} + B_{Y_R} = (n+2-d)^2 + d - 1$ , which allow us to compute the maximal eigenvalue of  $H_{1,n}$ ,

$$\max(H_{1,n}) = \frac{d-1}{n}. \quad (24)$$

Therefore, the desired conclusion follows.  $\square$

The above proof, along with the appendix results, provides a constructive algorithm for determining the  $m$ - $n$

TABLE I. Exact results for  $n$ - $m$  sharability of Werner states for different subsystem dimension, with  $m$  and  $n$  increasing from left to right and from top to bottom in each table, respectively. For each sharability setting, the value  $-\Phi$  is given. Asterisks corresponds to entries whose values have not been explicitly computed.

(a) $d = 2$						(b) $d = 3$						(c) $d = 4$					
$n, m$	1	2	3	4	5	$n, m$	1	2	3	4	5	$n, m$	1	2	3	4	5
1	1	1/2	1/3	1/4	1/5	1	1	1	2/3	1/2	2/5	1	1	1	1	3/4	3/5
2	1/2	1/2	1/3	1/4	1/5	2	1	1/2	1/2	1/2	2/5	2	1	1	2/3	1/2	1/2
3	1/3	1/3	1/3	1/4	1/5	3	2/3	1/2	1/3	1/3	1/3	3	1	2/3	5/9	1/2	*
4	1/4	1/4	1/4	1/4	1/5	4	1/2	1/2	1/3	1/4	1/4	4	3/4	1/2	1/2	*	*
5	1/5	1/5	1/5	1/5	1/5	5	2/5	1/3	1/3	1/4	*	5	3/5	1/2	*	*	*

sharability of Werner states. Stated again, the most-entangled  $m$ - $n$  sharable Werner state corresponds to the largest eigenvalue of  $H_{m,n}$ . Appendix B details the method of obtaining these values using Young diagrams. Although we lack a general closed-form expression for

$\max(H_{m,n})$ , the required calculation can nevertheless be performed numerically. Representative results for  $n$ - $m$  sharability of low-dimensional Werner states are shown in Table I.

The simplicity of the result in Eq. (23) is intriguing and begs for an intuitive interpretation. Consider a central qudit surrounded by  $n$  outer qudits. If the central qudit is in the same Werner state with each outer qudit, then Theorem III.3, rewritten as

$$-n\Phi \leq (d-1),$$

states that the sum of all the central-to-outer concurrences cannot exceed the number of modes by which the systems may disagree. This rule does not hold in more general joining scenarios, as we already know from Sec. IIIB. There, we found that the trade-off between  $A$ - $B$  concurrence and  $A$ - $C$  concurrence is not a linear one, as such a simple “sum rule” would predict; instead, it traces out an ellipse (recall Fig. 3).

#### IV. FURTHER REMARKS

##### A. Sharability off the Werner line

While our analysis has focused on a specific class of bipartite quantum states, some conclusions can be drawn for states not belonging to the Werner family. For the latter,  $1$ - $n$  sharability clearly implies  $n$ - $n$  sharability, by virtue of their symmetry. This property, however, does not hold in general:

**Proposition IV.1.** *For a generic bipartite qudit state  $\rho$ ,  $1$ - $n$  sharability does not imply  $n$ - $n$  sharability.*

*Proof.* It suffices to construct a counter example. We claim that the following bipartite state on two qubits,

$$\rho = \frac{1}{3}[(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) + |10\rangle\langle 10|] \equiv \rho_{L_1 R_1},$$

is  $1$ - $2$  sharable but *not*  $2$ - $2$  sharable. To show that it is  $1$ - $2$  sharable, observe that any tripartite state  $w_3 \equiv |\psi_\theta\rangle\langle\psi_\theta|$  is a valid sharing state, with

$$|\psi_\theta\rangle \equiv \frac{1}{\sqrt{3}}(|000\rangle + |101\rangle + e^{i\theta}|110\rangle).$$

The above state may be equivalently written as

$$|\psi_\theta\rangle = \frac{1}{\sqrt{3}}|0\rangle \otimes |00\rangle + \sqrt{\frac{2}{3}}|1\rangle \otimes \frac{1}{\sqrt{2}}(|01\rangle + e^{i\theta}|10\rangle).$$

In order for  $\rho$  to be  $2$ - $2$  sharable, a four-partite state  $w_4$  must exist, such that  $\text{Tr}_{\hat{L}_i \hat{L}_j}(w_4) = \rho$ , for  $i, j = 1, 2$ . Any state which  $2$ - $2$  shares  $\rho$  must then  $1$ - $2$  share the pure entangled state  $w_3$ . That is, in constructing the  $2$ - $2$  sharing state for  $\rho$ , we bring in a fourth system  $L_2$  which must reduce (by tracing over  $L_1$  or  $L_2$ ) to  $w_3$ . But, since  $w_3$  is a pure entangled state, it is not sharable. Thus, there cannot exist a  $2$ - $2$  sharing state for  $\rho$ .  $\square$

In addition, our results allow us to put bounds (though not necessarily tight ones) on the sharability of *any* bipartite qudit state. It suffices to observe that any bipartite state can be transformed into a Werner state by the action of the twirling map. Theorem II.9 proves that the sharability of a state cannot be decreased by a unital LOCC map, and hence twirling cannot decrease sharability. This thus establishes the following:

**Corollary IV.2.** *A bipartite qudit state  $\rho$  is no more sharable than the Werner state*

$$\tilde{\rho} \equiv \int U \otimes U \rho_V U^\dagger \otimes U^\dagger d\mu(U),$$

for any  $\rho_V = \mathbb{I} \otimes V \rho \mathbb{I} \otimes V^\dagger$ , with  $V \in \mathfrak{U}(d)$ .



In the qubit case, for instance, any maximally entangled pure state can be transformed into the singlet state by the action of some local unitary  $\mathbb{I} \otimes V$ . Thus, all maximally entangled pure qubit states and their “pseudo-pure” versions, obtained as mixtures with the fully mixed states, have the same sharability properties as the Werner states or, in other words, the pseudo-pure singlet states.

## B. Operational implications

As mentioned in the Introduction, from an *operational* point of view one would like to quantify how limited sharability implies or translates into a resource for some practical quantum tasks. While addressing this question requires a dedicated analysis which is beyond our current scope, an obvious quantum task which exploits entanglement is the standard teleportation protocol [46]. We show here that our sharability results can be used to obtain the limitations on a more general version of quantum teleportation known as “telecloning” [47]. While no new results are obtained, the optimality of  $1 \rightarrow n$  qubit telecloning naturally emerges as a corollary of our sharability results. Specifically, we have:

**Corollary IV.3.** *The optimal fidelity for  $1 \rightarrow n$  qubit telecloning is given by*

$$F(n) = \frac{2n+1}{3n}.$$

*Proof.* As shown in [48], in the optimal telecloning scenario, the bipartite reduced states from the  $(1+n)$ -party resource state are, without loss of optimality, Werner states. It then follows that the optimal resource state is the  $1-n$  sharing state for which each bipartite reduced state is made as Werner-entangled as possible. The determination of such a state and the Werner parameter of its bipartite reduced states is the content of Theorem III.3. Following [49], the fidelity of teleportation using a Werner state with parameter  $\Phi$  is given by  $F(\Phi) = \frac{2-\Phi}{3}$  for  $\Phi \leq 0$ , and  $F(\Phi) = \frac{2}{3}$  for  $\Phi \geq 0$ . Since, from Eq. (23),  $\Phi_{opt} = -1/n$ , the desired result follows.  $\square$

There are two generalizations to the  $1 \rightarrow n$  qubit telecloning scheme that are suggested by our general sharability results: qudit systems and cloning from  $m$  to  $n$  subsystems. In fact, Murao calls attention to both of these problems in [47]. In a later paper [50], an answer to the question of optimal telecloning from 1 to  $n$   $d$ -dimensional systems is obtained. The question of optimal standard cloning (as opposed to telecloning) of  $m$  copies of a qudit state to  $n$  copies of that state is addressed in both [51] and [52] (see also [53]).

Unfortunately, the utility of our sharability results, as applied to the optimal telecloning problem, is limited to the case of  $d = 2$ . Due to their rather large impurity and hence a limited fully entangled fraction [54], qudit Werner states with  $d > 2$  do not outperform the classical

limit in teleportation. Interestingly however, *isotropic states* [55], another one-parameter family of bipartite states, are provably optimal for the teleportation of qudit states in any dimension. Analogous to the invariance properties of Werner states under  $U \otimes U$  transformations, isotropic states have  $U \otimes U^*$  symmetry, making their analysis a foreseeably tractable expansion of the work herein. Thus, we expect an analysis of the sharability of such states to provide answers to the optimal telecloning of general  $d$ -dimensional systems.

## V. CONCLUSIONS

We have provided a general framework for defining the notions of quantum joinability and sharability in multipartite quantum systems, and compared both to the analogous classical notions. Special emphasis has been given to identifying the role of entanglement in both scenarios. In order to obtain mathematically necessary and sufficient conditions, we have specifically analyzed the  $1-2$  joinability and the  $m-n$  sharability properties of qudit Werner states. For the former, we found that the entanglement content of each of the two relevant bipartite states does *not* suffice to determine the resulting joinability properties. For the latter, we found a simple analytical expression for  $1-n$  sharing, namely, that the sum of the bipartite concurrences cannot exceed  $d - 1$ . In the more general case of  $m-n$  sharing, we laid out an algorithmic procedure for calculating the most-entangled  $m-n$  sharable Werner states using Young tableaux. As corollaries of our Werner-state results, we both established upper bounds on the sharability of *any* bipartite qudit state, and recovered the optimal fidelity of the  $1-n$  qubit telecloning protocol.

Several open questions remain for future investigations. Keeping with the approach pursued here, further progress toward obtaining necessary and sufficient joinability conditions may be made by narrowing the set of states to be analyzed to other physically relevant families and/or by considering specific joining structures. On the one hand, as mentioned in Sec. IV, isotropic qudit states are a natural extension for investigation in light of their general usefulness in  $d$ -dimensional teleportation and telecloning protocols. It is also worth noting that isotropic states map to the depolarizing quantum channel under the Choi-Jamiołkowski isomorphism [56, 57]. Thus, results on the joinability of isotropic states are additionally expected to be linked to the problem of *joining quantum channels*, which offers a rich area for exploration on its own. On the other hand, families of mixed qudit states may arise as reduced states of many-body ground states of spin-1/2 or higher-spin Hamiltonians parametrized by an external control parameter. In this context, it may be insightful to examine how joinability and sharability of quantum correlations change as the system is driven across a quantum phase transition, complementing extensive investigation of ground-state entan-

lement [58] and generalized entanglement [59, 60] in critical phenomena. Finally, since generalized entanglement is in fact defined without relying on a preferred tensor-product decomposition, with “generalized reduced states” being constructed through a suitable reduction map relative to observable subspaces [59, 61], a natural question arises: What is the nature and role of joinability limitations *beyond subsystems*? We leave exploration of this intriguing question to future research.

## ACKNOWLEDGMENTS

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## Appendix A: Non-negativity Characterization of Tripartite Werner States

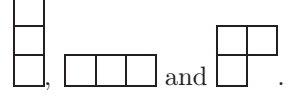
Joinability limitations are a manifestation of the non-negativity constraint placed on the joining density operators. Thus, a simple expression for non-negativity grants us a simple expression for joinability limitations. In considering the joinability of Werner states, we are interested in the non-negativity boundary in the space of collective-unitary-invariant  $N$ -qudit operators. This space is spanned by the set of subsystem permutation operators,

$$w = \sum_{\pi \in S_N} \mu_\pi V_\pi. \quad (\text{A1})$$

The above parameterization does not offer a straightforward characterization of non-negativity. However, we can choose a different basis for which non-negativity is more simply expressed in terms of its (the basis’) coefficients. The key idea is to decompose the operator space into subspaces for which the non-negativity of a given operator’s projection into each subspace ensures the non-negativity of the given operator. This is achieved if the operators within each operator subspace act non-trivially on orthogonal vector subspaces. Irreducible representations provide such a decomposition.

For  $S_N$ , the irrep subspaces are projected into by the so called Young symmetrizers. Each Young symmetrizer may be labelled by a Young diagram with a number of boxes equal to the number of subsystems. In the case we are interested in, there are three possible Young diagrams,

namely,



A Young tableau is obtained by placing the numbers 1 through  $N$  into the  $N$  boxes. A Young tableau is called standard if the numbers in each row and column are increasing from the upper left box. The rank of each corresponding Young symmetrizer is given by the number of standard Young tableau supported by that diagram. In the above case, the ranks are 1, 1, and 2, respectively. We find congruence with our parameterization since the dimensions of the operator spaces acting on these irreducible subspaces are  $1^2$ ,  $1^2$ , and  $2^2$ , summing to 6.

The prescription for constructing Young symmetrizers from the permutation representations may be found in most books on representation theory (e.g. [42],[62]). In our case the Young symmetrizers are

$$\begin{aligned} R_+ &= \frac{1}{6}(\mathbb{I} + V_{(AB)} + V_{(BC)} + V_{(CA)} + V_{(ABC)} + V_{(CBA)}), \\ R_- &= \frac{1}{6}(\mathbb{I} - V_{(AB)} - V_{(BC)} - V_{(CA)} + V_{(ABC)} + V_{(CBA)}), \\ R_0 &= \frac{1}{3}(2\mathbb{I} - V_{(ABC)} - V_{(CBA)}), \end{aligned} \quad (\text{A2})$$

and an orthonormal basis of operators acting on the support of  $R_0$  is

$$\begin{aligned} R_1 &= \frac{1}{3}(2V_{(BC)} - V_{(CA)} - V_{(AB)}), \\ R_2 &= \frac{1}{\sqrt{3}}(V_{(AB)} - V_{(CA)}), \\ R_3 &= \frac{i}{\sqrt{3}}(V_{(ABC)} - V_{(CBA)}). \end{aligned} \quad (\text{A3})$$

Defining  $r_k(w) = \text{Tr}(wR_k)$ , any  $w \in \mathcal{W}$  is of the form

$$\begin{aligned} w(\vec{r}) &= \frac{6r_+}{d(d^2 + 3d + 2)}R_+ + \frac{6r_-}{d(d^2 - 3d + 2)}R_- \\ &\quad + \frac{3}{2d(d^2 - 1)}\sum_{i=0}^3 r_i R_i, \end{aligned} \quad (\text{A4})$$

where  $\vec{r} = (r_+, r_-, \dots, r_3)$ . Normalization is ensured by

$$\text{Tr}(w\mathbb{I}) = \text{Tr}(w(R_+R_- + R_0)) = r_+ + r_- + r_0 = 1,$$

whereas non-negativity is given by the following simple relationships:

$$r_+, r_-, r_0 \geq 0, \quad r_1^2 + r_2^2 + r_3^2 \leq r_0^2.$$

## Appendix B: Joint Eigenvalues of Casimir Operators

In Sec. III C we make use of the eigenvalue structure of quadratic Casimir operators on tensor product representations of  $\mathfrak{su}(d)$ . In the case of  $\mathfrak{su}(2)$ , relations among the

Casimir eigenvalues are obtained by the familiar practice of angular momentum addition. Given an algebra with an orthonormal basis  $\{\lambda^\alpha\}$ , the quadratic Casimir operator is defined as

$$\Lambda^2 = \sum_{\alpha} \lambda^\alpha \lambda^\alpha \equiv \vec{\lambda} \cdot \vec{\lambda}.$$

Though it is not formally an element of the algebra,  $\Lambda^2$  commutes with all elements of the algebra thanks to the properties of the algebraic product. The Casimir operator on an irreducible module is a multiple of the identity.

Given a representation of an algebra on a single module  $V$ , we can construct another representation of that algebra on the  $N$ -fold tensor product  $V \otimes V \otimes \dots \otimes V$ . Each basis element  $\lambda^\alpha$  of the single module representation corresponds to a basis element  $\sum_{i \in \mathbb{Z}_N} \mathbb{I}_1 \otimes \dots \otimes \lambda_i^\alpha \otimes \dots \otimes \mathbb{I}_N$  of the  $N$ -module representation. A tensor product representation is in general reducible. That is, we can find subspaces of  $V \otimes V \otimes \dots \otimes V$  which are invariant under the action of the product algebra, and that contain no further invariant subspaces. Such a subspace, seen as a module, is said to be an irreducible representation of the algebra and we may decompose a direct product of modules into a direct sum,

$$V \otimes V \otimes \dots \otimes V = W_1 \oplus W_2 \oplus \dots \oplus W_k.$$

Schur-Weyl duality states that if the modules  $V$  provide a representation of the general linear group  $\mathfrak{gl}(d)$ , then the irreducible subspaces  $W_i$  are those subspaces projected into by the Young symmetrizers; in other words, each irreducible subspace, with respect to the  $N$ -fold action of  $\mathfrak{gl}(d)$ , is labelled by a Young diagram.

Seen as a valid algebra, any tensor product algebra supports its own quadratic Casimir operator defined just as before,

$$\begin{aligned} \Lambda_N^2 &= \sum_{\alpha} \sum_{i \in \mathbb{Z}_N} \mathbb{I}_1 \otimes \dots \otimes \lambda_i^\alpha \cdot \lambda_i^\alpha \otimes \dots \otimes \mathbb{I}_N \\ &\equiv \vec{\lambda}_N \cdot \vec{\lambda}_N. \end{aligned}$$

To make contact with the physics problem we are after, we show that  $H_{m,n}$  can be written in terms of Casimir operators. Rewriting each swap operator as  $V_{ij} = \mathbb{I}/d + \sum_{\alpha} \lambda_i^\alpha \lambda_j^\alpha$ , we obtain

$$\begin{aligned} H_{m,n} &= -\frac{\mathbb{I}}{d} - \frac{1}{mn} \sum_{i \in L} \sum_{j \in R} \sum_{\alpha} \lambda_i^\alpha \lambda_j^\alpha \\ &= -\frac{\mathbb{I}}{d} - \frac{1}{2mn} \left[ \sum_{i,j \in L \cup R} \vec{\lambda}_i \cdot \vec{\lambda}_j \right. \\ &\quad \left. - \sum_{i,j \in L} \vec{\lambda}_i \cdot \vec{\lambda}_j - \sum_{i,j \in R} \vec{\lambda}_i \cdot \vec{\lambda}_j \right] \\ &= \frac{1}{2mn} (\Lambda_L^2 + \Lambda_R^2 - \Lambda_{LR}^2) - \frac{\mathbb{I}}{d}, \end{aligned} \quad (\text{B1})$$

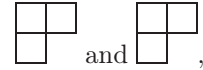
where we have dropped and will continue to drop the  $\mathbb{I}$  for the sake of brevity.

We note two important features of tensor product Casimir operators. First,  $\Lambda_N^2$  will not simply be proportional to the identity operator, but rather, on each irreducible subspace  $W_i$  it acts as a (possibly) different multiple of identity. Secondly, the operator  $\Lambda_N^2 \otimes \mathbb{I}_M$  commutes with  $\Lambda_{N+M}^2$  for any  $M$  and  $N$ . This is because any irreducible subspace of  $W_i$  of  $\Lambda_{N+M}^2$  labelled by a Young diagram will be contained in an irreducible subspace of  $W_i$  of  $\Lambda_N^2$  once tensored with  $\mathbb{I}_M$ .

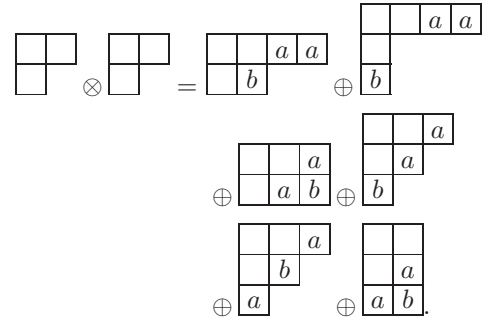
We are interested in the case of the three Casimir operators  $\Lambda_{LR}^2$ ,  $\Lambda_L^2$ , and  $\Lambda_R^2$ . These operators mutually commute and so we can seek their simultaneous eigenvalues. Each eigenvalue of a tensor product Casimir operator corresponds to an irreducible subspace  $W_i$  and hence to a Young diagram. The latter may be used to compute the value of the corresponding eigenvalue. Following [43], given a Young diagram  $Y$  of column heights  $\{a_i\}$  and row lengths  $\{b_j\}$ , the eigenvalue of the corresponding space of the Casimir operator is

$$C_Y = N \left( d - \frac{N}{d} \right) + \sum_j b_j^2 - \sum_i a_i^2. \quad (\text{B2})$$

The eigenspaces of  $\Lambda_{LR}^2$  which intersect with a given  $\Lambda_L^2$  eigenspace and a  $\Lambda_R^2$  eigenspace may be calculated by pasting together the boxes of a  $Y_L$  and a  $Y_R$  Young tableau in a way that does not cause two previously symmetrized boxes (same row) to then be anti-symmetrized (same column) and vice versa. For example, given the Young diagrams



we can construct the following composite diagrams



Writing  $A_Y = \sum_i a_i^2$  and  $B_Y = \sum_j b_j^2$ , and replacing each Casimir operator in Eq. B1 with its eigenvalue given by Eq. B2 we find that the sum of the first terms of the eigenvalues cancels with  $\mathbb{I}/d$ , leaving,

$$\text{eig}(H_{m,n}) = \frac{A_{Y_{LR}} - A_{Y_L} - A_{Y_R} - B_{Y_{LR}} + B_{Y_L} + B_{Y_R}}{2mn}.$$

Thus, we have a prescription for calculating the eigenvalues of  $H_{m,n}$ . We obtain the maximal eigenvalue of  $H_{m,n}$ , as is sought in Sec. III, by constructing the optimal set of three Young diagrams.

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